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1998 J. Phys. A: Math. Gen. 31 5241

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J. Phys. A: Math. Gen. **31** (1998) 5241–5262. Printed in the UK PII: S0305-4470(98)88873-2

Integrabilities of the t–J model with impurities

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Received 30 October 1997

Abstract. The Hamiltonian with magnetic impurities coupled to the strongly correlated electron system is constructed from the t–J model. It is diagonalized exactly by using the Bethe ansatz method. Our boundary matrices depend on the spins of the electrons. The Kondo problem in this system is discussed in detail. The integral equations are derived with complex rapidities which describe the bound states in the system. The finite-size corrections for the ground-state energies are obtained.

1. Introduction

The Kondo problem, devoted to studying the effects due to the exchange interaction between the impurity spin and electron gas, has played an important role in condensed matter physics since its discovery [1]. In the original treatments of the Kondo problem, the electron–electron interaction is discarded. This is reasonable in three dimensions where the interacting electron system can be described by a Fermi liquid. Recently, much attention has been paid to the theory of the magnetic impurities in the Fermi liquid and Luttinger liquid [2, 3] where the impurities are coupled to strongly correlated electron system. Apart from the fundamental theoretical interests, it is remarkable that the physics implied here can be accessible experimentally. The recent advances in semiconductor technology enable us to fabricate very narrow quantum wire which can be considered to be one dimensional (1D) and furnishes a real system of Luttinger liquid. Also, edge states in a two-dimensional (2D) electron gas for fractional quantum Hall effects can be considered as a Luttinger liquid [4]. Intense efforts and much progress has been made around the subjects from different approaches. Using bosonization and renormalization techniques, Kane and Fisher [5] studied transport of a 1D interacting electron due to potential barriers. Their results triggered the study of the problem of local perturbations to a Luttinger liquid and Kondo problem in a Luttinger liquid. The Kondo problem in a Luttinger liquid was considered by Lee and Toner [6]. They also performed the renormalization group calculation and found the crossover of the Kondo temperature from a power law dependence on the Kondo coupling constant to an exponential one. Relying on the poor man's scaling method, Furusaki and

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0305-4470/98/235241+22$19.50 © 1998 IOP Publishing Ltd

Nagaosa [7] showed that the Kondo coupling flows to the strong-coupling regime not only for the antiferromagnetic case but also for the ferromagnetic case. The boundary conformal field theory [8] allows a classification of critical behaviour for a Luttinger liquid coupled to a magnetic impurity. It turns out that there are two possibilities, a local Fermi liquid with standard low-temperature thermodynamics or a non-Fermi liquid [7]. The non-Fermi liquid behaviour is induced by the tunnelling effect of conduction electrons through the impurity which depends only on the bulk properties, not on the details of the impurity [9]. Density matrix renormalization group calculation also supports the same conclusion [10]. In addition, the renormalization group flow diagram for parameters characterizing impurity is more complex and contains fixed points responsible for the low-temperature behaviours when the potential of impurity is not strong [11].

Despite all important progress hitherto made, the problem of few impurities embedded in a strongly correlated 1D electron system is still far from a complete understanding. We think that exact solutions of some integrable models on the subjects are useful from which one can expect to draw definite conclusions. Indeed Bedürftig *et al* [12] has thoroughly solved an integrable model with impurity coupled with the t–J chain. They introduced the impurity through local vertices as in [13]. The model introduced has a lack of backward scattering and the presence of redundant terms in the Hamiltonian. Based on Kane and Fisher's observation [5], we see that it is advantageous to use an open boundary problem with the impurities at open ends to study the problem of impurities coupled with strongly correlated electron systems. The programme has been initiated for a δ-interacting Fermi system in [14] for the t–J model in [15] and for the Hubbard model in [16].

The t–J model, is considered to be one of the most fundamental models in strongly correlated electron systems because of its possible relevance for purely electronic mechanisms for high-T_c superconductivity and heavy-fermion systems. This model is obtained from the Hubbard model as an effective Hamiltonian for the low-energy states in the strong-correlation limit. In this limit double occupancy of fermions is forbidden, leading to only three possible states at each lattice site for half spin. Currently, there is a greater demand for its study. Very recently, the Luttinger liquid properties of the t–J model were discussed in [17]. By solving the functional relations, the finite-size corrections related to the t–J model were dealt with for the open boundary conditions in [18]. The effects of an integrable impurity coupling to both spin and charge degrees of freedom are studied in a periodic t–J chain [12] which we have mentioned above. Another generalization of the t–J model is given in [19] where they used the one-parametric family of four-dimensional representations of $gl(2|1)$. It is also a kind of generalization of the extended Hubbard model [20].

In this paper we expand the study of the Kondo problem in the 1D t–J model [15] by an exact solution of open boundary Bethe ansatz. For this purpose we put two magnetic impurities in both sides of the open t–J model which is a typical situation for the 1D systems with impurities. The coupling constants of the impurities with conduction electrons cover from negative infinity to positive infinity, which means that both the ferromagnetic Kondo effect and antiferromagnetic Kondo effect can be dealt with on the same setting. We then construct the Hamiltonian for the system with magnetic impurities from the t–J model. The integrability of this model ensures that both the Yang–Baxter equation and the reflecting Yang–Baxter equation are satisfied. By using the algebraic Bethe ansatz scheme for open boundary [21] we diagonalize the Hamiltonian for the present system and obtain the Bethe ansatz equations. From which we derive the nonlinear integral equations governing the thermodynamic properties of the model for large system. The finite-size corrections for energy of ground states in all cases can be calculated.

This paper is organized as follows. In section 2 the constructed Hamiltonian and its first quantization form are given explicitly. In section 3 the boundary matrix depending on the rapidity and spin of the particle is given and all possible integrable cases for the model are exhausted. The Bethe ansatz equations of the systems for all integrable cases are derived in section 4. The properties of the ground state for the cases other than that in [15] are discussed in great detail in section 5. In the final section the finite-size corrections of the ground-state energies for particular cases are obtained.

2. The Hamiltonian of the model

Consider a 1D lattice with G sites, N electrons and two magnetic impurities at both ends. Due to a large on-site Coulomb repulsion there is at most one particle at each site. The dynamics of the system are governed by a Hamiltonian which we construct from the t–J model [22–27]. The conduction electrons can hop (t) between the neighbouring sites. There are four types interactions in the model. A spin exchange interaction J and a charge interaction independent of the spin of strength V between the neighbour conduction electron; Kondo coupling J_a, J_b and impurity potential interactions V_a, V_b between the electron and impurities. The Hamiltonian of the system has the form:

$$H = -t\sum_{j=1}^{G-1}\sum_{\sigma=\uparrow\downarrow}(C^+_{j\sigma}C_{j+1\sigma} + C^+_{j+1\sigma}C_{j\sigma}) + J\sum_{j=1}^{G-1}\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}$$
$$+V\sum_{j=1}^{G-1}n_jn_{j+1} + J_a\boldsymbol{S}_1\cdot\boldsymbol{S}_a + V_an_1 + J_b\boldsymbol{S}_G\cdot\boldsymbol{S}_b + V_bn_G \qquad (1)$$

where $C^+_{j\sigma}(C_{j\sigma})$ is the creation (annihilation) operator of the conduction electron with spin σ on the site j; $J_{a,b}$, $V_{a,b}$ are the Kondo coupling constants and the impurity potentials, respectively; $\boldsymbol{S}_j = \frac{1}{2}\sum_{\sigma,\sigma'}C^+_{j\sigma}\sigma_{\sigma,\sigma'}C_{j\sigma'}$ is the spin operator of the conduction electron; $n_j = C^+_{j\uparrow}C_{j\uparrow} + C^+_{j\downarrow}C_{j\downarrow}$ is the number operator of the conduction electron; G is the length (or site number) of the system. Some properties of the ground state for $t = 1$, $J = 2$, $V = \frac{3}{2}$ were reported in [15]. Following Schultz's notation [28] we write the translation operators $T^\pm_j$:

$$T^\pm_j\Psi(x_1,\ldots,x_j,\ldots,x_N) = \Psi(x_1,\ldots,x_j\pm 1,\ldots,x_N)$$

where $\Psi(x_1,\ldots,x_j,\ldots,x_N)$ is the wavefunction of N conduction electrons. In first quantization and in appropriate energy units ($t = 1$) the Hamiltonian of this system can be written as

$$H = -\sum_{j=1}^{N}(T^+_j + T^-_j) + \sum_{j=1}^{N}(K_{aj}\delta_{x_j,1} + K_{bj}\delta_{x_j,G} + K_j) \qquad (2)$$

where the couplings are denoted by operators $K_{aj} = V_a - \frac{J_a}{4} + \frac{J_a}{2}P_{aj}$ and $K_{bj} = V_b - \frac{J_b}{4} + \frac{J_b}{2}P_{bj}$ with the permutation operators $P_{a(b),j}$ between the spins of the conduction electron j and the impurities a (b). The operator K_j acts on the wavefunction Ψ as

$$K_j\Psi(x_1,x_2,\ldots,x_N) = \sum_{i=1}^{N}\delta_{x_j,x_i+1}K_{ij}\Psi(x_1,x_2,\ldots,x_N)$$

where $K_{ij} = V - \frac{J}{4} + \frac{J}{2}P_{ij}$ describes the interactions between the conduction electrons with the permutation operator P_{ij} permuting ith and jth electrons in spin space. We will diagonalize the above Hamiltonian in the following section.

3. Integrability conditions

We write the wavefunction in region $0 \leqslant x_{Q1} \leqslant x_{Q2} \leqslant \cdots \leqslant x_{QN} \leqslant L-1$ as

$$\Psi_{\sigma_1,\sigma_2,\ldots,\sigma_N}(x_1, x_2, \ldots, x_N) = \sum_P \sum_{r_1,r_2,\ldots r_N=\pm1} \varepsilon_P \varepsilon_r A_{\sigma_{Q1},\sigma_{Q2},\ldots,\sigma_{QN}} (r_{PQ1}k_{PQ1}, r_{PQ2}k_{PQ2}, \ldots, r_{PQN}k_{PQN}) \exp\left[\mathrm{i}\sum_{j=1}^{N} r_{Pj}k_{Pj}x_j\right] \tag{3}$$

where the coefficients $A_{\sigma_{Q1},\sigma_{Q2},\ldots,\sigma_{QN}}(r_{PQ1}k_{PQ1}, r_{PQ2}k_{PQ2}, \ldots, r_{PQN}k_{PQN})$ are also dependent on the spins of magnetic impurities which are suppressed for brevity, and $\varepsilon_P = 1$ (-1), when the parity of P is even (odd), $\varepsilon_r = \prod_{j=1}^N r$ in which r takes the value $+1$ or -1. The boundary R-matrix satisfies the reflecting Yang–Baxter equation:

$$S_{12}(\lambda,\mu) \overset{1}{R}(\lambda) S_{12}(\lambda,-\mu) \overset{2}{R}(\mu) = \overset{2}{R}(\mu) S_{12}(\lambda,-\mu) \overset{1}{R}(\lambda) S_{12}(\lambda,\mu) \tag{4}$$

where operators $\overset{1}{R}(\lambda)$ and $\overset{2}{R}(\mu)$ are defined as

$$\overset{1}{R}(\lambda) = R(\lambda)\otimes \mathrm{id}_{V_2} \qquad \overset{2}{R}(\mu) = \mathrm{id}_{V_1}\otimes R(\mu)$$

for matrix $R \in \mathrm{End}(V)$. The S-matrix satisfies the normal factorizable condition:

$$S_{12}(k,\lambda)S_{13}(k,\mu)S_{23}(\lambda,\mu) = S_{23}(\lambda,\mu)S_{13}(k,\mu)S_{12}(k,\lambda). \tag{5}$$

For convenience we set $t = 1$. From the reflecting Yang–Baxter equation and the form for S-matrix, the boundary R-matrix should have the form

$$R = \exp(\mathrm{i}\varphi)\frac{q - \mathrm{i}C - \mathrm{i}P}{q + \mathrm{i}C + \mathrm{i}P} \tag{6}$$

where P is the permutation operator, $q = \pm\frac{1}{2}\cot\frac{k}{2}, \pm\frac{1}{2}\tan\frac{k}{2}$ and C is the arbitrary constant. Putting $K_{a(b),j} = m + lP$, we have from equation (4) that

$$q[(m-1)^2 - l^2]\tan^2\frac{k}{2} + 2l(q^2 + C^2 - 1)\tan\frac{k}{2} + q[(m+1)^2 - l^2] = 0. \tag{7}$$

This is the restriction imposed on coupling constants for our model (1) to be integrable. The details are as follows.

$J = 2$, $V = -\frac{1}{2}$, in this case we know that the scattering matrix in the bulk can be written as:

$$S_{12}(k_1,k_2) = \frac{\frac{1}{2}\cot\frac{k_1}{2} - \frac{1}{2}\cot\frac{k_2}{2} - \mathrm{i}P_{12}}{\frac{1}{2}\cot\frac{k_1}{2} - \frac{1}{2}\cot\frac{k_2}{2} - \mathrm{i}} \tag{8}$$

where P_{12} is the permutation operator between two electrons. The boundary R-matrix at the left end of the chain takes the form:

$$R_a(k_j,\sigma_j) = \exp[\mathrm{i}\varphi_a(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}P_{aj}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}P_{aj}}. \tag{9}$$

The coupling constants J_a, V_a at the left end of the chain are expressed in terms of C_a

$$J_a = -\frac{8}{(2C_a \mp 1)(2C_a \pm 3)} \qquad V_a = \frac{3 - 4C_a^2}{(2C_a \mp 1)(2C_a \pm 3)} \tag{10}$$

and

$$\exp\left[\mathrm{i}\varphi_a(k_j)\right] = \frac{J_a(\cot\frac{k_j}{2} + 2\mathrm{i}C_a)\exp(\mathrm{i}k_j) + \mathrm{i}[4 + (4V_a - J_a)\exp(\mathrm{i}k_j)]}{J_a(\cot\frac{k_j}{2} - 2\mathrm{i}C_a)\exp(-\mathrm{i}k_j) - \mathrm{i}[4 + (4V_a - J_a)\exp(-\mathrm{i}k_j)]}. \tag{11}$$

The boundary R-matrix at the right end of the chain has the form:

$$R_b(-k_j, \sigma_j) = \exp[-2\mathrm{i}k_j(G+1) + \mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}P_{bj}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}P_{bj}}. \tag{12}$$

Similar relations exist for J_b, V_b and $\varphi_b(k_j)$ by merely substituting indices a into (10) and (11) by b.

3.1. $J = -2$, $V = \frac{1}{2}$

The boundary R-matrices have the forms:

$$R_a(k_j, \sigma_j) = \exp[\mathrm{i}\varphi_a(k_j)]\frac{\frac{1}{2}\tan\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}P_{aj}}{\frac{1}{2}\tan\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}P_{aj}} \tag{13}$$

$$R_b(-k_j, \sigma_j) = \exp[-2\mathrm{i}k_j(G+1) + \mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\tan\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}P_{bj}}{\frac{1}{2}\tan\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}P_{bj}}. \tag{14}$$

where $\varphi_a(k_j)$ and $\varphi_b(k_j)$ are the same as in the proceeding. Now the coupling constants should be written in terms of the arbitrary parameter C_a in the form

$$J_a = \frac{8}{(2C_a \mp 1)(2C_a \pm 3)} \qquad V_a = \frac{4C_a^2 - 3}{(2C_a \mp 1)(2C_a \pm 3)}. \tag{15}$$

J_b, V_b have the same expressions except with the substitution of indices a by b. Correspondingly, the scattering matrix S in the bulk for two conduction electrons is

$$S_{12}(k_1, k_2) = \frac{\frac{1}{2}\tan\frac{k_1}{2} - \frac{1}{2}\tan\frac{k_2}{2} + \mathrm{i}P_{12}}{\frac{1}{2}\tan\frac{k_1}{2} - \frac{1}{2}\tan\frac{k_2}{2} + \mathrm{i}}. \tag{16}$$

3.2. $J = 2$, $V = \frac{3}{2}$

In this case the dependence of coupling constants on parameter C_a takes the form

$$J_a = -\frac{8}{(2C_a \mp 1)(2C_a \pm 3)} \qquad V_a = \frac{4C_a^2 - 7}{(2C_a \mp 1)(2C_a \pm 3)}. \tag{17}$$

J_b, V_b have the same expressions by the substituting of indices a by b. The scattering matrix in the bulk is

$$S_{12}(k_1, k_2) = -\frac{\frac{1}{2}\tan\frac{k_1}{2} - \frac{1}{2}\tan\frac{k_2}{2} - \mathrm{i}P_{12}}{\frac{1}{2}\tan\frac{k_1}{2} - \frac{1}{2}\tan\frac{k_2}{2} + \mathrm{i}}. \tag{18}$$

The boundary R-matrices are

$$R_a(k_j, \sigma_j) = \exp[\mathrm{i}\varphi_a(k_j)]\frac{\frac{1}{2}\tan\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}P_{aj}}{\frac{1}{2}\tan\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}P_{aj}} \tag{19}$$

$$R_b(-k_j, \sigma_j) = \exp[-2\mathrm{i}k_j(G+1) + \mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\tan\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}P_{bj}}{\frac{1}{2}\tan\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}P_{bj}}. \tag{20}$$

3.3. $J = -2$, $V = -\frac{3}{2}$

The coupling constants have the forms

$$J_a = \frac{8}{(2C_a \mp 1)(2C_a \pm 3)} \qquad V_a = \frac{7 - 4C_a^2}{(2C_a \mp 1)(2C_a \pm 3)}. \tag{21}$$

J_b, V_b have the same expressions by the substituting of indices a by b. The scattering matrix in the bulk is

$$S_{12}(k_1, k_2) = -\frac{\frac{1}{2}\cot\frac{k_1}{2} - \frac{1}{2}\cot\frac{k_2}{2} + \mathrm{i}P_{12}}{\frac{1}{2}\cot\frac{k_1}{2} - \frac{1}{2}\cot\frac{k_2}{2} - \mathrm{i}}. \tag{22}$$

The boundary R-matrices are

$$R_a(k_j, \sigma_j) = \exp[\mathrm{i}\varphi_a(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}P_{aj}}{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}P_{aj}} \tag{23}$$

$$R_b(-k_j, \sigma_j) = \exp[-2\mathrm{i}k_j(G+1) + \mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}P_{bj}}{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}P_{bj}}. \tag{24}$$

The expressions for boundary matrices depending on both the moment of the particle and the spin of the electron are new. The expressions of the S-matrix in the bulk have already been obtained in [22], but they are different from ours.

4. Bethe ansatz equations

By using the standard Bethe ansatz procedure, we can diagonalize the Hamiltonian (1) [21] and obtain the following Bethe ansatz equations. When $J = 2$ and $V = -\frac{1}{2}$, setting

$$S_{j0}(k_j, k_0) = \frac{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_0}{2} - \mathrm{i}P_{j0}}{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_0}{2} - \mathrm{i}} \tag{25}$$

$$S_{jN+1}(k_j, k_{N+1}) = \frac{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_{N+1}}{2} - \mathrm{i}P_{jN+1}}{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_{N+1}}{2} - \mathrm{i}} \tag{26}$$

where $\cot\frac{k_0}{2} = 2\mathrm{i}C_a$, $\cot\frac{k_{N+1}}{2} = 2\mathrm{i}C_b$, $P_{j0} \equiv P_{aj}$, $P_{jN+1} \equiv P_{bj}$, we can write the boundary R-matrices as the forms:

$$R_a(k_j, \sigma_j) = \exp[\mathrm{i}\varphi_a(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}}\frac{S_{j0}(k_j, k_0)}{S_{j0}(-k_j, k_0)} \tag{27}$$

$$R_b(-k_j, \sigma_j) = \exp[-2\mathrm{i}k_j(G+1) + \mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}}\frac{S_{jN+1}(k_j, k_{N+1})}{S_{jN+1}(-k_j, k_{N+1})}. \tag{28}$$

Define

$$T(\lambda) = S_{\tau j}(\lambda)S_{\tau 0}(\lambda)S_{\tau 1}(\lambda)\ldots S_{\tau j-1}(\lambda)S_{\tau j+1}(\lambda)\ldots S_{\tau N}(\lambda)S_{\tau N+1}(\lambda) \tag{29}$$

with

$$S_{\tau l}(\lambda) = \frac{\lambda - \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}P_{\tau l}}{\lambda - \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}} \qquad l = 0, 1, \ldots, N+1. \tag{30}$$

We obtain the equation

$$\mathrm{Tr}[T(\lambda)T^{-1}(-\lambda)]\big|_{\lambda=\frac{1}{2}\cot\frac{k_j}{2}}\Phi = \frac{2\mathrm{i}-\cot\frac{k_j}{2}}{\mathrm{i}-\cot\frac{k_j}{2}}\frac{\frac{1}{2}\cot\frac{k_j}{2}+\mathrm{i}C_a+\mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2}-\mathrm{i}C_a-\mathrm{i}}\frac{\frac{1}{2}\cot\frac{k_j}{2}+\mathrm{i}C_b+\mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2}-\mathrm{i}C_b-\mathrm{i}} \times \exp[-\mathrm{i}\varphi_a(k_j)-\mathrm{i}\varphi_b(k_j)+2\mathrm{i}k_j(G+1)]\Phi \tag{31}$$

where Φ is the eigenstate of the system. Then the Bethe ansatz equations can be expressed as

$$\exp[2\mathrm{i}k_j(G+1)-\mathrm{i}\varphi_a(k_j)-\mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2}+\mathrm{i}C_a+\mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2}-\mathrm{i}C_a-\mathrm{i}}\frac{\frac{1}{2}\cot\frac{k_j}{2}+\mathrm{i}C_b+\mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2}-\mathrm{i}C_b-\mathrm{i}} = \prod_{\beta=1}^{M}\frac{\frac{1}{2}\cot\frac{k_j}{2}-\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2}-\lambda_\beta-\frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\cot\frac{k_j}{2}+\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2}+\lambda_\beta-\frac{\mathrm{i}}{2}} \qquad (j=1,2,\ldots,N) \tag{32}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_a^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_a^2}\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_b^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_b^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-\frac{1}{2}\cot\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha-\frac{1}{2}\cot\frac{k_l}{2}-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+\frac{1}{2}\cot\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha+\frac{1}{2}\cot\frac{k_l}{2}-\frac{\mathrm{i}}{2}} = \prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}}\frac{\lambda_\alpha+\lambda_\beta+\mathrm{i}}{\lambda_\alpha+\lambda_\beta-\mathrm{i}} \qquad (\alpha=1,2,\ldots,M). \tag{33}$$

M is the number of down spins and N is the number of electrons. The function φ is denoted by expression (11). Similarly, when $J=-2$ and $V=\frac{1}{2}$, we can write down the Bethe ansatz equations as the forms:

$$\exp[2\mathrm{i}k_j(G+1)-\mathrm{i}\varphi_a(k_j)-\mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_a-\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_a+\mathrm{i}}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_b-\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_b+\mathrm{i}} = \prod_{\beta=1}^{M}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta-\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta+\frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta-\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta+\frac{\mathrm{i}}{2}} \qquad (j=1,2,\ldots,N) \tag{34}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_a^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_a^2}\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_b^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_b^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}} = \prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}}\frac{\lambda_\alpha+\lambda_\beta+\mathrm{i}}{\lambda_\alpha+\lambda_\beta-\mathrm{i}} \qquad (\alpha=1,2,\ldots,M). \tag{35}$$

When $J=2$ and $V=\frac{3}{2}$, we have that

$$\exp[2\mathrm{i}k_j(G+1)-\mathrm{i}\varphi_a(k_j)-\mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_a+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_a-\mathrm{i}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_b+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_b-\mathrm{i}} \times\prod_{l=1(l\neq j)}^{N}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\frac{1}{2}\tan\frac{k_l}{2}+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}-\frac{1}{2}\tan\frac{k_l}{2}-\mathrm{i}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\frac{1}{2}\tan\frac{k_l}{2}+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}+\frac{1}{2}\tan\frac{k_l}{2}-\mathrm{i}} = \prod_{\beta=1}^{M}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta-\frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta-\frac{\mathrm{i}}{2}} \qquad (j=1,2,\ldots,N) \tag{36}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_a^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_a^2}\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_b^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_b^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}} = \prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}}\frac{\lambda_\alpha+\lambda_\beta+\mathrm{i}}{\lambda_\alpha+\lambda_\beta-\mathrm{i}} \qquad (\alpha=1,2,\ldots,M). \tag{37}$$

When $J = -2$ and $V = -\frac{3}{2}$, we obtain that

$$\exp[2\mathrm{i}k_j(G+1) - \mathrm{i}\varphi_a(k_j) - \mathrm{i}\varphi_b(k_j)]\frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}}\frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}}$$
$$\times \prod_{l=1(l\neq j)}^{N} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_l}{2} + \mathrm{i}}\frac{\frac{1}{2}\cot\frac{k_j}{2} + \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \frac{1}{2}\cot\frac{k_l}{2} + \mathrm{i}}$$
$$= \prod_{\beta=1}^{M} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta - \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta + \frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta - \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta + \frac{\mathrm{i}}{2}} \qquad (j = 1, 2, \ldots, N) \tag{38}$$

$$\frac{(\lambda_\alpha + \frac{\mathrm{i}}{2})^2 + C_a^2}{(\lambda_\alpha - \frac{\mathrm{i}}{2})^2 + C_a^2}\frac{(\lambda_\alpha + \frac{\mathrm{i}}{2})^2 + C_b^2}{(\lambda_\alpha - \frac{\mathrm{i}}{2})^2 + C_b^2}\prod_{l=1}^{N}\frac{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}\frac{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}$$
$$= \prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}}\frac{\lambda_\alpha + \lambda_\beta + \mathrm{i}}{\lambda_\alpha + \lambda_\beta - \mathrm{i}} \qquad (\alpha = 1, 2, \ldots, M). \tag{39}$$

Here the function $\varphi_a(k_{j)}$ is expressed by equation (11) and $\varphi_b(k_j)$ has the same expression as relation (11) with the substitution of index a by b. M is the number of down spins and N is the number of the electrons. It should be noted that in the above Bethe ansatz equations we have chosen the boundary R-matrices to have the same form as those in section 3, where the boundary matrices depend on the spin parameter. If the R-matrix is dependent on the spin of the electron only at one end of the chain, for example, we denote by $R_b(k_j, \hat{\sigma}_j)$ the boundary matrix at the right end of the chain independent on the spin σ_j. The Bethe ansatz equations for $J = 2$, $V = -\frac{1}{2}$ take the form:

$$\frac{\exp[-\mathrm{i}\varphi_a(k_j)]}{R_b(-k_j, \hat{\sigma}_j)}\frac{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}} = \prod_{\beta=1}^{M}\frac{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta + \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta - \frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta + \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta - \frac{\mathrm{i}}{2}}$$
$$(j = 1, 2, \ldots, N) \tag{40}$$

$$\frac{(\lambda_\alpha + \frac{\mathrm{i}}{2})^2 + C_a^2}{(\lambda_\alpha - \frac{\mathrm{i}}{2})^2 + C_a^2}\prod_{l=1}^{N}\frac{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}\frac{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}$$
$$= \prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}}\frac{\lambda_\alpha + \lambda_\beta + \mathrm{i}}{\lambda_\alpha + \lambda_\beta - \mathrm{i}} \qquad (\alpha = 1, 2, \ldots, M). \tag{41}$$

Similarly, when the boundary matrix at the left end of the spin is independent of the spin of the electron, denoted by $R_a(k_j, \hat{\sigma}_j)$, we have that

$$\frac{\exp[2\mathrm{i}k_j(G+1) - \mathrm{i}\varphi_b(k_j)]}{R_a(k_j, \hat{\sigma}_j)}\frac{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}} = \prod_{\beta=1}^{M}\frac{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta + \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta - \frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta + \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta - \frac{\mathrm{i}}{2}}$$
$$(j = 1, 2, \ldots, N) \tag{42}$$

$$\frac{(\lambda_\alpha + \frac{\mathrm{i}}{2})^2 + C_a^2}{(\lambda_\alpha - \frac{\mathrm{i}}{2})^2 + C_a^2}\prod_{l=1}^{N}\frac{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}\frac{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}$$
$$= \prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}}\frac{\lambda_\alpha + \lambda_\beta + \mathrm{i}}{\lambda_\alpha + \lambda_\beta - \mathrm{i}} \qquad (\alpha = 1, 2, \ldots, M) \tag{43}$$

where the number of down spins should be less than $N+2$ and N is the number of the conduction electrons in the system. Furthermore, we obtain that

$$\frac{\exp[-\mathrm{i}\varphi_a(k_j)]}{R_b(-k_j,\hat{\sigma}_j)}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_a-\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_a+\mathrm{i}}=\prod_{\beta=1}^{M}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta-\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta+\frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta-\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta+\frac{\mathrm{i}}{2}}$$
$$(j=1,2,\ldots,N) \tag{44}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_a^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_a^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}$$
$$=\prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}}\frac{\lambda_\alpha+\lambda_\beta+\mathrm{i}}{\lambda_\alpha+\lambda_\beta-\mathrm{i}}\qquad(\alpha=1,2,\ldots,M) \tag{45}$$

and

$$\frac{\exp[2\mathrm{i}k_j(G+1)-\mathrm{i}\varphi_b(k_j)]}{R_a(k_j,\hat{\sigma}_j)}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_b-\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_b+\mathrm{i}}=\prod_{\beta=1}^{M}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta-\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta+\frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta-\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta+\frac{\mathrm{i}}{2}}$$
$$(j=1,2,\ldots,N) \tag{46}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_b^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_b^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}$$
$$=\prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}}\frac{\lambda_\alpha+\lambda_\beta+\mathrm{i}}{\lambda_\alpha+\lambda_\beta-\mathrm{i}}\qquad(\alpha=1,2,\ldots,M) \tag{47}$$

for the case of $J=-2$, $V=\frac{1}{2}$.

$$\frac{\exp[-\mathrm{i}\varphi_a(k_j)]}{R_b(-k_j,\hat{\sigma}_j)}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_a+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_a-\mathrm{i}}\prod_{l=1(l\neq j)}^{N}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\frac{1}{2}\tan\frac{k_l}{2}+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}-\frac{1}{2}\tan\frac{k_l}{2}-\mathrm{i}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\frac{1}{2}\tan\frac{k_l}{2}+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}+\frac{1}{2}\tan\frac{k_l}{2}-\mathrm{i}}$$
$$=\prod_{\beta=1}^{M}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta-\frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta-\frac{\mathrm{i}}{2}}\qquad(j=1,2,\ldots,N) \tag{48}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_a^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_a^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}$$
$$=\prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}}\frac{\lambda_\alpha+\lambda_\beta+\mathrm{i}}{\lambda_\alpha+\lambda_\beta-\mathrm{i}}\qquad(\alpha=1,2,\ldots,M) \tag{49}$$

and

$$\frac{\exp[2\mathrm{i}k_j(G+1)-\mathrm{i}\varphi_b(k_j)]}{R_a(k_j,\hat{\sigma}_j)}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\mathrm{i}C_b+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}-\mathrm{i}C_b-\mathrm{i}}$$
$$\times\prod_{l=1(l\neq j)}^{N}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\frac{1}{2}\tan\frac{k_l}{2}+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}-\frac{1}{2}\tan\frac{k_l}{2}-\mathrm{i}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\frac{1}{2}\tan\frac{k_l}{2}+\mathrm{i}}{\frac{1}{2}\tan\frac{k_j}{2}+\frac{1}{2}\tan\frac{k_l}{2}-\mathrm{i}}$$
$$=\prod_{\beta=1}^{M}\frac{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}-\lambda_\beta-\frac{\mathrm{i}}{2}}\frac{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta+\frac{\mathrm{i}}{2}}{\frac{1}{2}\tan\frac{k_j}{2}+\lambda_\beta-\frac{\mathrm{i}}{2}}\qquad(j=1,2,\ldots,N) \tag{50}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_b^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_b^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha-\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}+\frac{\mathrm{i}}{2}}{\lambda_\alpha+\frac{1}{2}\tan\frac{k_l}{2}-\frac{\mathrm{i}}{2}}$$

$$= \prod_{\beta=1(\beta\neq\alpha)}^{M} \frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}} \frac{\lambda_\alpha + \lambda_\beta + \mathrm{i}}{\lambda_\alpha + \lambda_\beta - \mathrm{i}} \qquad (\alpha = 1, 2, \ldots, M) \tag{51}$$

for the case of $J = 2$, $V = \frac{3}{2}$ when boundary matrix only at one end of the chain rely on the spin parameter of the electron. Finally, for the case of $J = -2$ and $V = -\frac{3}{2}$, the Bethe ansatz equations take the form:

$$\frac{\exp[-\mathrm{i}\varphi_a(k_j)]}{R_b(-k_j, \hat{\sigma}_j)} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_a - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_a + \mathrm{i}} \prod_{l=1(l\neq j)}^{N} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_l}{2} + \mathrm{i}} \frac{\frac{1}{2}\cot\frac{k_j}{2} + \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \frac{1}{2}\cot\frac{k_l}{2} + \mathrm{i}}$$

$$= \prod_{\beta=1}^{M} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta - \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta + \frac{\mathrm{i}}{2}} \frac{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta - \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta + \frac{\mathrm{i}}{2}} \qquad (j = 1, 2, \ldots, N) \tag{52}$$

$$\frac{(\lambda_\alpha + \frac{\mathrm{i}}{2})^2 + C_a^2}{(\lambda_\alpha - \frac{\mathrm{i}}{2})^2 + C_a^2} \prod_{l=1}^{N} \frac{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}} \frac{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}$$

$$= \prod_{\beta=1(\beta\neq\alpha)}^{M} \frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}} \frac{\lambda_\alpha + \lambda_\beta + \mathrm{i}}{\lambda_\alpha + \lambda_\beta - \mathrm{i}} \qquad (\alpha = 1, 2, \ldots, M) \tag{53}$$

and

$$\frac{\exp[2\mathrm{i}k_j(G+1) - \mathrm{i}\varphi_b(k_j)]}{R_a(k_j, \hat{\sigma}_j)} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \mathrm{i}C_b - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \mathrm{i}C_b + \mathrm{i}}$$

$$\times \prod_{l=1(l\neq j)}^{N} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} - \frac{1}{2}\cot\frac{k_l}{2} + \mathrm{i}} \frac{\frac{1}{2}\cot\frac{k_j}{2} + \frac{1}{2}\cot\frac{k_l}{2} - \mathrm{i}}{\frac{1}{2}\cot\frac{k_j}{2} + \frac{1}{2}\cot\frac{k_l}{2} + \mathrm{i}}$$

$$= \prod_{\beta=1}^{M} \frac{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta - \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} - \lambda_\beta + \frac{\mathrm{i}}{2}} \frac{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta - \frac{\mathrm{i}}{2}}{\frac{1}{2}\cot\frac{k_j}{2} + \lambda_\beta + \frac{\mathrm{i}}{2}} \qquad (j = 1, 2, \ldots, N) \tag{54}$$

$$\frac{(\lambda_\alpha + \frac{\mathrm{i}}{2})^2 + C_b^2}{(\lambda_\alpha - \frac{\mathrm{i}}{2})^2 + C_b^2} \prod_{l=1}^{N} \frac{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha - \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}} \frac{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} + \frac{\mathrm{i}}{2}}{\lambda_\alpha + \frac{1}{2}\cot\frac{k_l}{2} - \frac{\mathrm{i}}{2}}$$

$$= \prod_{\beta=1(\beta\neq\alpha)}^{M} \frac{\lambda_\alpha - \lambda_\beta + \mathrm{i}}{\lambda_\alpha - \lambda_\beta - \mathrm{i}} \frac{\lambda_\alpha + \lambda_\beta + \mathrm{i}}{\lambda_\alpha + \lambda_\beta - \mathrm{i}} \qquad (\alpha = 1, 2, \ldots, M). \tag{55}$$

$R_a(k_j, \hat{\sigma}_j)$ and $R_b(-k_j, \hat{\sigma}_j)$ denote that the boundary matrices at the left and the right ends of the system are independent on the spin σ_j respectively. Note that the number of down spins is less than $N + 2$ for the system with N conduction electrons. In the following section, we focus the discussions on the system with the boundary matrices depending on the spins of the electrons at both ends of the chain. Set

$$\begin{aligned}\theta_a(k) &= \frac{1}{\mathrm{i}} \ln \frac{(4C_a^2 - 3)\cos k - 4C_a^2 + 5 + 4\mathrm{i}C_a \sin k}{(4C_a^2 - 3)\cos k - 4C_a^2 + 5 - 4\mathrm{i}C_a \sin k} \\ \theta_b(k) &= \frac{1}{\mathrm{i}} \ln \frac{(4C_b^2 - 3)\cos k - 4C_b^2 + 5 + 4\mathrm{i}C_b \sin k}{(4C_b^2 - 3)\cos k - 4C_b^2 + 5 - 4\mathrm{i}C_b \sin k}.\end{aligned} \tag{56}$$

From relation (11) and

$$\exp[\mathrm{i}\varphi_b(k_j)] = \frac{J_b(\cot\frac{k_j}{2} + 2\mathrm{i}C_b)\exp(\mathrm{i}k_j) + \mathrm{i}[4 + (4V_b - J_b)\exp(\mathrm{i}k_j)]}{J_b(\cot\frac{k_j}{2} - 2\mathrm{i}C_b)\exp(-\mathrm{i}k_j) - \mathrm{i}[4 + (4V_b - J_b)\exp(-\mathrm{i}k_j)]} \tag{57}$$

we obtain the following expressions. When $J = 2$ and $V = -\frac{1}{2}$, we have that

$$\exp[\mathrm{i}\varphi_a(k)] = \begin{cases} \exp(\mathrm{i}k) & \text{for } J_a = -\dfrac{8}{(2C_a-1)(2C_a+3)}, \\ & \quad V_a = \dfrac{3-4C_a^2}{(2C_a-1)(2C_a+3)} \\ \exp[\mathrm{i}k + \mathrm{i}\theta_a(k)] & \text{for } J_a = -\dfrac{8}{(2C_a+1)(2C_a-3)}, \\ & \quad V_a = \dfrac{3-4C_a^2}{(2C_a+1)(2C_a-3)}. \end{cases} \tag{58}$$

When $J = -2$ and $V = \frac{1}{2}$, we have that

$$\exp[\mathrm{i}\varphi_a(k)] = \begin{cases} -\exp(\mathrm{i}k) & \text{for } J_a = \dfrac{8}{(2C_a-1)(2C_a+3)}, \\ & \quad V_a = \dfrac{4C_a^2-3}{(2C_a-1)(2C_a+3)} \\ -\exp[\mathrm{i}k - \mathrm{i}\theta_a(\pi-k)] & \text{for } J_a = \dfrac{8}{(2C_a+1)(2C_a-3)}, \\ & \quad V_a = \dfrac{4C_a^2-3}{(2C_a+1)(2C_a-3)}. \end{cases} \tag{59}$$

When $J = 2$ and $V = \frac{3}{2}$, we have that

$$\exp[\mathrm{i}\varphi_a(k)] = \begin{cases} -\exp[\mathrm{i}k + \mathrm{i}\theta_a(\pi-k)] & \text{for } J_a = \dfrac{-8}{(2C_a-1)(2C_a+3)}, \\ & \quad V_a = \dfrac{4C_a^2-7}{(2C_a-1)(2C_a+3)} \\ -\exp(\mathrm{i}k) & \text{for } J_a = \dfrac{-8}{(2C_a+1)(2C_a-3)}, \\ & \quad V_a = \dfrac{4C_a^2-7}{(2C_a+1)(2C_a-3)}. \end{cases} \tag{60}$$

When $J = -2$ and $V = -\frac{3}{2}$, we have that

$$\exp[\mathrm{i}\varphi_a(k)] = \begin{cases} \exp[\mathrm{i}k + \mathrm{i}\theta_a(-k)] & \text{for } J_a = \dfrac{8}{(2C_a-1)(2C_a+3)}, \\ & \quad V_a = \dfrac{7-4C_a^2}{(2C_a-1)(2C_a+3)} \\ \exp(\mathrm{i}k) & \text{for } J_a = \dfrac{8}{(2C_a+1)(2C_a-3)}, \\ & \quad V_a = \dfrac{7-4C_a^2}{(2C_a+1)(2C_a-3)}. \end{cases} \tag{61}$$

The expressions of $\exp[\mathrm{i}\varphi_b(k)]$ can be obtained by substituting the index a of b in the above relations. Then, without any loss of generalization, we can choose that

$$\begin{aligned} J_a &= -\frac{8}{(2C_a-1)(2C_a+3)} \qquad & V_a &= \frac{3-4C_a^2}{(2C_a-1)(2C_a+3)} \\ J_b &= -\frac{8}{(2C_b-1)(2C_b+3)} \qquad & V_b &= \frac{3-4C_b^2}{(2C_b-1)(2C_b+3)} \end{aligned} \tag{62}$$

for $J=2$ and $V=-\frac{1}{2}$. The Bethe ansatz equations take the forms as

$$\frac{q_{j+\mathrm{i}(C_a+1)}}{q_{j-\mathrm{i}(C_a+1)}}\frac{q_{j+\mathrm{i}(C_b+1)}}{q_{j-\mathrm{i}(C_b+1)}}\exp(2\mathrm{i}k_jG)=\prod_{\beta=1}^{M}\frac{q_j-\lambda_\beta+\frac{\mathrm{i}}{2}}{q_j-\lambda_\beta-\frac{\mathrm{i}}{2}}\frac{q_j+\lambda_\beta+\frac{\mathrm{i}}{2}}{q_j+\lambda_\beta-\frac{\mathrm{i}}{2}} \tag{63}$$

$$\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_a^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_a^2}\frac{(\lambda_\alpha+\frac{\mathrm{i}}{2})^2+C_b^2}{(\lambda_\alpha-\frac{\mathrm{i}}{2})^2+C_b^2}\prod_{l=1}^{N}\frac{\lambda_\alpha-q_l+\frac{\mathrm{i}}{2}}{\lambda_\alpha-q_l-\frac{\mathrm{i}}{2}}\frac{\lambda_\alpha+q_l+\frac{\mathrm{i}}{2}}{\lambda_\alpha+q_l-\frac{\mathrm{i}}{2}}$$
$$=\prod_{\beta=1(\beta\neq\alpha)}^{M}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}}\frac{\lambda_\alpha+\lambda_\beta+\mathrm{i}}{\lambda_\alpha+\lambda_\beta-\mathrm{i}} \tag{64}$$

where $q_j=\frac{1}{2}\cot\frac{k_j}{2}$. For the case of $J=-2$ and $V=\frac{1}{2}$, the Bethe ansatz equations have also the forms (63) and (64) with $q_j=-\frac{1}{2}\tan\frac{k_j}{2}$ and the Kondo coupling constants should be

$$\begin{aligned} J_a&=\frac{8}{(2C_a-1)(2C_a+3)} \qquad V_a=\frac{4C_a^2-3}{(2C_a-1)(2C_a+3)}\\ J_b&=\frac{8}{(2C_b-1)(2C_b+3)} \qquad V_b=\frac{4C_b^2-3}{(2C_b-1)(2C_b+3)}. \end{aligned} \tag{65}$$

If $J=2$ and $V=\frac{3}{2}$, we choose that

$$\begin{aligned} J_a&=-\frac{8}{(2C_a+1)(2C_a-3)} \qquad V_a=\frac{4C_a^2-7}{(2C_a+1)(2C_a-3)}\\ J_b&=-\frac{8}{(2C_b+1)(2C_b-3)} \qquad V_b=\frac{4C_b^2-7}{(2C_b+1)(2C_b-3)} \end{aligned} \tag{66}$$

and the Bethe ansatz equations are

$$\exp(2\mathrm{i}k_jG)\frac{q_{j+\mathrm{i}(C_a+1)}}{q_{j-\mathrm{i}(C_a+1)}}\frac{q_{j+\mathrm{i}(C_b+1)}}{q_{j-\mathrm{i}(C_b+1)}}\prod_{l=1(l\neq j)}^{N}\frac{q_j-q_l+\mathrm{i}}{q_j-q_l-\mathrm{i}}\frac{q_j+q_l+\mathrm{i}}{q_j+q_l-\mathrm{i}}$$
$$=\prod_{\beta=1}^{M}\frac{q_j-\lambda_\beta+\frac{\mathrm{i}}{2}}{q_j-\lambda_\beta-\frac{\mathrm{i}}{2}}\frac{q_j+\lambda_\beta+\frac{\mathrm{i}}{2}}{q_j+\lambda_\beta-\frac{\mathrm{i}}{2}} \tag{67}$$

and relation (64) with $q_j=\frac{1}{2}\tan\frac{k_j}{2}$. They are also the Bethe ansatz equations for $J=-2$ and $V=-\frac{3}{2}$ with $q_j=-\frac{1}{2}\cot\frac{k_j}{2}$ and

$$\begin{aligned} J_a&=\frac{8}{(2C_a+1)(2C_a-3)} \qquad V_a=\frac{7-4C_a^2}{(2C_a+1)(2C_a-3)}\\ J_b&=\frac{8}{(2C_b+1)(2C_b-3)} \qquad V_b=\frac{7-4C_b^2}{(2C_b+1)(2C_b-3)}. \end{aligned} \tag{68}$$

5. Ground state

In this paper we restrict the discussions of the properties of ground state to the case of $J=\pm2$ and $V=\mp\frac{1}{2}$. The cases of $J=2$ and $V=\frac{3}{2}$ were studied in [15]. The eigenvalue of the Hamiltonian is

$$E=\mp2N\pm\sum_{j=1}^{N}\frac{1}{q_j^2+\frac{1}{4}} \tag{69}$$

for $J = 2$, $V = -\frac{1}{2}$ with $q_j = \frac{1}{2}\cot\frac{k_j}{2}$ and $J = -2$, $V = \frac{1}{2}$ with $q_j = -\frac{1}{2}\tan\frac{k_j}{2}$, respectively. They satisfy the Bethe ansatz equations (63) and (64) from which the integral equations are derived.

5.1. Integral equations

Following [29], we introduce the notation

$$e(x) \equiv \frac{x+\mathrm{i}}{x-\mathrm{i}}.$$

Then, from relations (63) and (64) we obtain

$$e\left(\frac{q_j}{1+C_a}\right)e\left(\frac{q_j}{1+C_b}\right)e(2q_j)^{2G} = \prod_{\beta=1}^{M} e\left(2q_j - 2\lambda_\beta\right)e\left(2q_j + 2\lambda_\beta\right) \tag{70}$$

$$e\left(\frac{\lambda_\alpha}{\frac{1}{2}-C_a}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}+C_a}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}-C_b}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}+C_b}\right)\prod_{l=1}^{N} e(2\lambda_\alpha - 2q_l)e(2\lambda_\alpha + 2q_l)$$
$$= \prod_{\beta=1(\beta\neq\alpha)}^{M} e(\lambda_\alpha - \lambda_\beta)e(\lambda_\alpha + \lambda_\beta) \tag{71}$$

where $j = 1, 2, \ldots, N$; $\alpha = 1, 2, \ldots, M$ and $e(\pm\infty) = 1$. Considering that the parameter q_j can take complex values, the general structure for $\{q_j\}_{j=1,2,\ldots,N}$ should consist of M' pairs of $q_\alpha^{\pm} = \lambda_\alpha \pm \frac{\mathrm{i}}{2} + \mathrm{O}(\exp(-\delta G), \alpha = 1, \ldots, M'$ and M'' pairs of $\tilde{q}_\alpha^{\pm} = -\tilde{\lambda}_\alpha \pm \frac{\mathrm{i}}{2} + \mathrm{O}(\exp(-\delta G), \tilde{\lambda}_\alpha \in \{\lambda_\beta\}$ and remaining $N - 2(M' + M'')$ non-pairing q_j''s. To be more precise, we use

$$Q \equiv \{q_j \,|\, j = 1, 2, \ldots, N\} = X' \cup X'' \cup Y$$

where

$$\begin{aligned} X' &= \left\{q_\alpha^{\pm} = \lambda_\alpha \pm \frac{\mathrm{i}}{2} + \mathrm{O}(\exp(-\delta G) | \alpha = 1, \ldots, M'\right\} \\ X'' &= \left\{\tilde{q}_\alpha^{\pm} = -\tilde{\lambda}_\alpha \pm \frac{\mathrm{i}}{2} + \mathrm{O}(\exp(-\delta G) | \tilde{\lambda}_\alpha \in \{\lambda_\beta\}, \alpha = 1, \ldots, M''\right\} \\ Y &= Q \backslash (X' \cup X''). \end{aligned} \tag{72}$$

Obviously, the non-pairing q_j satisfies equation (70) with $j = 1, 2, \ldots, N - 2(M' + M'')$. When $q_j \in X'$, from equation (70), we have

$$e\left(\frac{\lambda_\alpha}{\frac{3}{2}+C_a}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}+C_a}\right)e\left(\frac{\lambda_\alpha}{\frac{3}{2}+C_b}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}+C_b}\right)e(\lambda_\alpha)^{2G}$$
$$= e(2q_\alpha^{+} - 2\lambda_\alpha)e(2q_\alpha^{-} - 2\lambda_\alpha)e(2\lambda_\alpha)$$
$$\times \prod_{\beta=1,(\beta\neq\alpha)}^{M} e(\lambda_\alpha - \lambda_\beta)e(\lambda_\alpha + \lambda_\beta) \qquad \alpha = 1, 2, \ldots, M'. \tag{73}$$

From equation (71) we have

$$e\left(\frac{\lambda_\alpha}{\frac{1}{2}-C_a}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}+C_a}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}-C_b}\right)e\left(\frac{\lambda_\alpha}{\frac{1}{2}+C_b}\right)e(2\lambda_\alpha)$$
$$\times \prod_{l=1}^{N-2M_-} e(2\lambda_\alpha - 2q_l)e(2\lambda_\alpha + 2q_l) \prod_{\beta=1}^{M-M_-} e(\hat{\lambda}_\beta - \lambda_\alpha)e(-\hat{\lambda}_\beta - \lambda_\alpha)$$

$$= e(2q_\alpha^+ - 2\lambda_\alpha)e(2q_\alpha^- - 2\lambda_\alpha) \tag{74}$$

where $M_- = M' + M''$ and $\hat{\lambda}_\beta$ ($\beta = 1, 2, \ldots, M - M_-$) are the parameters describing the down spins but having no contributions to the bound states. With the help of the above relation, equation (73) becomes

$$e\left(\frac{\lambda_\alpha}{C_a+\frac{3}{2}}\right)e\left(\frac{\lambda_\alpha}{C_a-\frac{1}{2}}\right)e\left(\frac{\lambda_\alpha}{C_b+\frac{3}{2}}\right)e\left(\frac{\lambda_\alpha}{C_b-\frac{1}{2}}\right)e(\lambda_\alpha)^{2G}$$
$$= e(2\lambda_\alpha)^2\prod_{l=1}^{N-2M_-} e(2\lambda_\alpha - 2q_l)e(2\lambda_\alpha + 2q_l)\prod_{\beta=1(\beta\neq\alpha)}^{M'} e(\lambda_\alpha - \lambda_\beta)e(\lambda_\alpha + \lambda_\beta)$$
$$\times\prod_{\beta=1}^{M''} e(\lambda_\alpha - \tilde{\lambda}_\beta)e(\lambda_\alpha + \tilde{\lambda}_\beta) \qquad \alpha = 1, \ldots, M'. \tag{75}$$

Similarly, when $q_j \in X''$, we obtain the following equation

$$e\left(\frac{\tilde{\lambda}_\alpha}{C_a+\frac{3}{2}}\right)e\left(\frac{\tilde{\lambda}_\alpha}{C_a-\frac{1}{2}}\right)e\left(\frac{\tilde{\lambda}_\alpha}{C_b+\frac{3}{2}}\right)e\left(\frac{\tilde{\lambda}_\alpha}{C_b-\frac{1}{2}}\right)e(\tilde{\lambda}_\alpha)^{2G}$$
$$= e(2\tilde{\lambda}_\alpha)^2\prod_{l=1}^{N-2M_-} e(2\tilde{\lambda}_\alpha - 2q_l)e(2\tilde{\lambda}_\alpha + 2q_l)\prod_{\beta=1}^{M'} e(\tilde{\lambda}_\alpha - \lambda_\beta)e(\tilde{\lambda}_\alpha + \lambda_\beta)$$
$$\times\prod_{\beta=1(\beta\neq\alpha)}^{M''} e(\tilde{\lambda}_\alpha - \tilde{\lambda}_\beta)e(\tilde{\lambda}_\alpha + \tilde{\lambda}_\beta) \qquad \alpha = 1, \ldots, M''. \tag{76}$$

The two equations (75) and (76) can be combined into a single equation

$$e\left(\frac{\lambda_\alpha}{C_a+\frac{3}{2}}\right)e\left(\frac{\lambda_\alpha}{C_a-\frac{1}{2}}\right)e\left(\frac{\lambda_\alpha}{C_b+\frac{3}{2}}\right)e\left(\frac{\lambda_\alpha}{C_b-\frac{1}{2}}\right)e(\lambda_\alpha)^{2G}$$
$$= e\,(2\lambda_\alpha)^2\prod_{l=1}^{N-2M_-} e(2\lambda_\alpha - 2q_l)e(2\lambda_\alpha + 2q_l)$$
$$\times\prod_{\beta=1(\beta\neq\alpha)}^{M_-} e(\lambda_\alpha - \lambda_\beta)e(\lambda_\alpha + \lambda_\beta) \qquad \alpha = 1, 2, \ldots, M_- \tag{77}$$

with the new λ_α defined by

$$\lambda_\alpha = \begin{cases} \lambda_\alpha & \text{when } \alpha = 1, 2, \ldots, M' \\ \tilde{\lambda}_{M'-\alpha} & \text{when } \alpha = M'+1, M'+2, \ldots, M_-. \end{cases}$$

The parameters $\hat{\lambda}_\alpha$ ($\alpha = 1, 2, \ldots, M - M_-$), in view of (74), satisfy

$$e\left(\frac{\hat{\lambda}_\alpha}{C_a+\frac{1}{2}}\right)e\left(\frac{\hat{\lambda}_\alpha}{C_b+\frac{1}{2}}\right)\prod_{l=1}^{N-2M_-} e(2\hat{\lambda}_\alpha - 2q_l)e(2\hat{\lambda}_\alpha + 2q_l)$$
$$= e\left(\frac{\hat{\lambda}_\alpha}{C_a-\frac{1}{2}}\right)e\left(\frac{\hat{\lambda}_\alpha}{C_b-\frac{1}{2}}\right)\prod_{\beta=1(\beta\neq\alpha)}^{M-M_-} e(\hat{\lambda}_\alpha - \hat{\lambda}_\beta)e(\hat{\lambda}_\alpha + \hat{\lambda}_\beta). \tag{78}$$

The non-pairing q_j (i.e. $q_j \in Y$) satisfies

$$e\left(\frac{q_j}{C_a+1}\right)e\left(\frac{q_j}{C_b+1}\right)e(2q_j)^{2G} = \prod_{\beta=1}^{M_-} e(2q_j - 2\lambda_\beta)e(2q_j + 2\lambda_\beta)$$

$$\times \prod_{\beta=1}^{M-M_-} e(2q_j - 2\hat{\lambda}_\beta)e(2q_j + 2\hat{\lambda}_\beta) \tag{79}$$

where $j = 1, 2, \ldots, N - 2M_-$ and $e(\pm\infty) = 1$. Setting

$$\theta(x) \equiv 2\tan^{-1} x \qquad -\pi < \theta \leqslant \pi$$

we have

$$e(x) = \exp[\mathrm{i}(\pi - \theta(x))].$$

The logarithms of equations (77)–(79) give, respectively,

$$\theta\left(\frac{\lambda_\alpha}{C_a + \frac{3}{2}}\right) + \theta\left(\frac{\lambda_\alpha}{C_a - \frac{1}{2}}\right) + \theta\left(\frac{\lambda_\alpha}{C_b + \frac{3}{2}}\right) + \theta\left(\frac{\lambda_\alpha}{C_b - \frac{1}{2}}\right) + 2G\theta(\lambda_\alpha)$$
$$= 4\pi J_\alpha + \theta(2\lambda_\alpha) + \sum_{l=1}^{N-2M_-}[\theta(2\lambda_\alpha - 2q_l) + \theta(2\lambda_\alpha + 2q_l)]$$
$$+ \sum_{\beta=1}^{M_-}[\theta(\lambda_\alpha - \lambda_\beta) + \theta(\lambda_\alpha + \lambda_\beta)] \tag{80}$$

with $\alpha = 1, 2, \ldots, M_-$ and integers or half-integer J_α;

$$\theta\left(\frac{\hat{\lambda}_\alpha}{C_a + \frac{1}{2}}\right) + \theta\left(\frac{\hat{\lambda}_\alpha}{C_b + \frac{1}{2}}\right) + \sum_{l=1}^{N-2M_-}[\theta(2\hat{\lambda}_\alpha - 2q_l) + \theta(2\hat{\lambda}_\alpha + 2q_l)]$$
$$= 4\pi \hat{J}_\alpha - \theta(2\lambda_\alpha) + \theta\left(\frac{\hat{\lambda}_\alpha}{C_a - \frac{1}{2}}\right) + \theta\left(\frac{\hat{\lambda}_\alpha}{C_b - \frac{1}{2}}\right)$$
$$+ \sum_{\beta=1}^{M-M_-}[\theta(\hat{\lambda}_\alpha - \hat{\lambda}_\beta) + \theta(\hat{\lambda}_\alpha + \hat{\lambda}_\beta)] \tag{81}$$

with $\alpha = 1, 2, \ldots, M - M_-$ and integers or half-integer $\hat{J}_\alpha$;

$$\theta\left(\frac{q_j}{C_a + 1}\right) + \theta\left(\frac{q_j}{C_b + 1}\right) + 2G\theta(2q_j) = 4\pi I_j + \sum_{\beta=1}^{M_-}[\theta(2q_j - 2\lambda_\beta) + \theta(2q_j + 2\lambda_\beta)]$$
$$+ \sum_{\beta=1}^{M-M_-}[\theta(2q_j - 2\hat{\lambda}_\beta) + \theta(2q_j + 2\hat{\lambda}_\beta)] \tag{82}$$

with $j = 1, 2, \ldots, N - 2M_-$and integers or half-integer I_j. By setting

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta[k(x + c)] = 2\pi a\left(x + c, \frac{1}{k}\right) \tag{83}$$

equations (80)–(82) can be changed into the forms

$$a(\lambda_\alpha, C_a + \tfrac{3}{2}) + a(\lambda_\alpha, C_a - \tfrac{1}{2}) + a(\lambda_\alpha, C_b + \tfrac{3}{2}) + a(\lambda_\alpha, C_b - \tfrac{1}{2}) + 2Ga(\lambda_\alpha, 1)$$
$$= \frac{2\mathrm{d}J_\alpha}{\mathrm{d}\lambda_\alpha} + a(\lambda_\alpha, \tfrac{1}{2}) + \sum_{l=1}^{N-2M_-}[a(\lambda_\alpha - q_l, \tfrac{1}{2}) + a(\lambda_\alpha + q_l, \tfrac{1}{2})]$$
$$+ \sum_{\beta=1}^{M_-}[a(\lambda_\alpha - \lambda_\beta, 1) + a(\lambda_\alpha + \lambda_\beta, 1)] \tag{84}$$

with $\alpha = 1, 2, \ldots, M_-$;

$$a(\hat{\lambda}_\alpha, C_a + \tfrac{1}{2}) + a(\hat{\lambda}_\alpha, C_b + \tfrac{1}{2}) + \sum_{l=1}^{N-2M_-} [a(\hat{\lambda}_\alpha - q_l, \tfrac{1}{2}) + a(\hat{\lambda}_\alpha + q_l, \tfrac{1}{2})]$$
$$= \frac{2\mathrm{d}\hat{J}_\alpha}{\mathrm{d}\hat{\lambda}_\alpha} - a(\hat{\lambda}_\alpha, \tfrac{1}{2}) + a(\hat{\lambda}_\alpha, C_a - \tfrac{1}{2}) + a(\hat{\lambda}_\alpha, C_b - \tfrac{1}{2})$$
$$+ \sum_{\beta=1}^{M-M_-} [a(\hat{\lambda}_\alpha - \hat{\lambda}_\beta, 1) + a(\hat{\lambda}_\alpha + \hat{\lambda}_\beta, 1)] \tag{85}$$

with $\alpha = 1, 2, \ldots, M - M_-$;

$$a(q_j, C_a + 1) + a(q_j, C_b + 1) + 2Ga(q_j, \tfrac{1}{2}) = \frac{2\mathrm{d}I_j}{\mathrm{d}q_j} + \sum_{\beta=1}^{M_-}[a(q_j - \lambda_\beta, \tfrac{1}{2}) + a(q_j + \lambda_\beta, \tfrac{1}{2})]$$
$$+ \sum_{\beta=1}^{M-M_-} [a(q_j - \hat{\lambda}_\beta, \tfrac{1}{2}) + a(q_j + \hat{\lambda}_\beta, \tfrac{1}{2})] \tag{86}$$

with $j = 1, 2, \ldots, N - 2M_-$. We define that

$$j(\lambda) \equiv \theta(\lambda) + \frac{1}{2G}\left\{\theta\left(\frac{\lambda}{C_a + \frac{3}{2}}\right) + \theta\left(\frac{\lambda}{C_a - \frac{1}{2}}\right) + \theta\left(\frac{\lambda}{C_b + \frac{3}{2}}\right) + \theta\left(\frac{\lambda}{C_b - \frac{1}{2}}\right) - \theta(2\lambda)\right\}$$
$$-\frac{1}{2G}\left\{\sum_{l=1}^{N-2M_-} [\theta(2\lambda - 2q_l) + \theta(2\lambda + 2q_l)] + \sum_{\beta=1}^{M_-}[\theta(\lambda - \lambda_\beta) + \theta(\lambda + \lambda_\beta)]\right\} \tag{87}$$

$$\hat{j}(\hat{\lambda}) \equiv \frac{1}{2G}\left\{\theta\left(\frac{\hat{\lambda}}{C_a + \frac{1}{2}}\right) - \theta\left(\frac{\hat{\lambda}}{C_a - \frac{1}{2}}\right) + \theta\left(\frac{\hat{\lambda}}{C_b + \frac{1}{2}}\right) - \theta\left(\frac{\hat{\lambda}}{C_b - \frac{1}{2}}\right) + \theta(2\hat{\lambda})\right\}$$
$$+\frac{1}{2G}\left\{\sum_{l=1}^{N-2M_-} [\theta(2\hat{\lambda} - 2q_l) + \theta(2\hat{\lambda} + 2q_l)]\right.$$
$$\left. - \sum_{\beta=1}^{M-M_-} [\theta(\hat{\lambda} - \hat{\lambda}_\beta) + \theta(\hat{\lambda} + \hat{\lambda}_\beta)]\right\} \tag{88}$$

$$h(q) \equiv \theta(2q) + \frac{1}{2G}\left\{\theta\left(\frac{q}{C_a + 1}\right) + \theta\left(\frac{q}{C_b + 1}\right)\right\}$$
$$-\frac{1}{2G}\left\{\sum_{\beta=1}^{M_-}[\theta(2q - 2\lambda_\beta) + \theta(2q + 2\lambda_\beta)]\right.$$
$$\left. + \sum_{\beta=1}^{M-M_-} [\theta(2q - 2\hat{\lambda}_\beta) + \theta(2q + 2\hat{\lambda}_\beta)]\right\}. \tag{89}$$

Then, the holes of $\lambda, \hat{\lambda}$ and q are defined as the solutions of

$$\begin{aligned} &Gj(\lambda) = 2\pi \times (\text{omitted } J) \\ &G\hat{j}(\hat{\lambda}) = 2\pi \times (\text{omitted } \hat{J}) \\ &Gh(q) = 2\pi \times (\text{omitted } I). \end{aligned} \tag{90}$$

By taking the thermodynamic limits, we introduce the distribution functions

$$\left.\begin{array}{l}\lambda\to\sigma(\lambda)\\ q\to\varrho(q)\\ \hat\lambda\to\hat\sigma(\hat\lambda)\end{array}\right\}\overset{\text{holes}}{\rightarrow}\left\{\begin{array}{l}\sigma^h(\lambda)\\ \varrho^h(q)\\ \hat\sigma^h(\hat\lambda).\end{array}\right.$$

So we have that

$$\begin{aligned}\frac{\mathrm{d}j(\lambda)}{\mathrm{d}\lambda}&=2\pi(\sigma(\lambda)+\sigma^h(\lambda))\\ \frac{\mathrm{d}h(q)}{\mathrm{d}q}&=2\pi(\rho(q)+\varrho^h(q))\\ \frac{\mathrm{d}\hat j(\hat\lambda)}{\mathrm{d}\hat\lambda}&=2\pi(\hat\sigma(\hat\lambda)+\hat\sigma^h(\hat\lambda)).\end{aligned}\tag{91}$$

Therefore, the integral equations can be written as

$$\begin{aligned}&2a(\lambda,1)+\frac{1}{G}[a(\lambda,C_a+\tfrac{3}{2})+a(\lambda,C_a-\tfrac{1}{2})+a(\lambda,C_b+\tfrac{3}{2})+a(\lambda,C_b-\tfrac{1}{2})]\\ &\quad=\frac{1}{G}a(\lambda,\tfrac{1}{2})+2\sigma(\lambda)+2\sigma^h(\lambda)+\int\mathrm{d}\lambda'\,\sigma(\lambda')[a(\lambda-\lambda',1)+a(\lambda+\lambda',1)]\\ &\qquad+\int\mathrm{d}q\,\rho(q)[a(\lambda-q,\tfrac{1}{2})+a(\lambda+q,\tfrac{1}{2})]\end{aligned}\tag{92}$$

$$\begin{aligned}&2a(q,\tfrac{1}{2})+\frac{1}{G}[a(q,C_a+1)+a(q,C_b+1)]=2\rho(q)+2\rho^h(q)\\ &\qquad+\int\mathrm{d}\lambda\,\sigma(\lambda)[a(q-\lambda,\tfrac{1}{2})+a(q+\lambda,\tfrac{1}{2})]\\ &\qquad+\int\mathrm{d}\hat\lambda\hat\sigma(\hat\lambda)[a(q-\hat\lambda,\tfrac{1}{2})+a(q+\hat\lambda,\tfrac{1}{2})]\end{aligned}\tag{93}$$

$$\begin{aligned}&\frac{1}{G}[a(\hat\lambda,\tfrac{1}{2})+a(\hat\lambda,C_a+\tfrac{1}{2})+a(\hat\lambda,C_b+\tfrac{1}{2})-a(\hat\lambda,C_a-\tfrac{1}{2})-a(\hat\lambda,C_b-\tfrac{1}{2})]\\ &\qquad+\int\mathrm{d}q\,\rho(q)[a(\hat\lambda-q,\tfrac{1}{2})+a(\hat\lambda+q,\tfrac{1}{2})]\\ &\quad=2\hat\sigma(\hat\lambda)+2\hat\sigma^h(\hat\lambda)+\int\mathrm{d}\hat\lambda'\,\hat\sigma(\hat\lambda')[a(\hat\lambda-\hat\lambda',1)+a(\hat\lambda+\hat\lambda',1)]\end{aligned}\tag{94}$$

where $a(\lambda,\eta)\equiv\eta/[\pi(\lambda^2+\eta^2)]$ with the arbitrary parameter η. The terms with factors of $1/G$ in the above three equations describe the finite-size corrections of the system.

5.2. Properties of ground state

For the system with N electrons, by using the distributed functions $\sigma(\lambda),\hat\sigma(\hat\lambda)$ and $\rho(q)$, the particle number and magnetization per unit length are given by

$$\begin{aligned}\frac{N}{G}&=\int\mathrm{d}q\,\rho(q)+2\int\mathrm{d}\lambda\,\sigma(\lambda)\\ \frac{S_z}{G}&=\frac{1}{2}\int\mathrm{d}q\,\rho(q)-\int\mathrm{d}\lambda\,\hat\sigma(\lambda).\end{aligned}\tag{95}$$

The energies per unit length have the forms as

$$\frac{E}{G}=-\frac{2N}{G}+2\pi\int\mathrm{d}q\,\rho(q)a(q,\tfrac{1}{2})+2\pi\int\mathrm{d}\lambda\,\sigma(\lambda)a(\lambda,1)\tag{96}$$

for the case of $J = 2$, $V = -\frac{1}{2}$ and

$$\frac{E}{G} = \frac{2N}{G} - 2\pi \int \mathrm{d}q\, \rho(q) a(q, \tfrac{1}{2}) - 2\pi \int \mathrm{d}\lambda\, \sigma(\lambda) a(\lambda, 1) \tag{97}$$

for the case of $J = -2$, $V = \frac{1}{2}$. Relations (92)–(94) become

$$2a(\lambda, 1) = 2\sigma(\lambda) + 2\sigma^h(\lambda) + \int \mathrm{d}\lambda'\, \sigma(\lambda')[a(\lambda - \lambda', 1) + a(\lambda + \lambda', 1)] \\ + \int \mathrm{d}q\, \rho(q)[a(\lambda - q, \tfrac{1}{2}) + a(\lambda + q, \tfrac{1}{2})] \tag{98}$$

$$2a(q, \tfrac{1}{2}) = 2\rho(q) + 2\rho^h(q) + \int \mathrm{d}\lambda\, \sigma(\lambda)[a(q - \lambda, \tfrac{1}{2}) + a(q + \lambda, \tfrac{1}{2})] \\ + \int \mathrm{d}\hat{\lambda}\, \hat{\sigma}(\hat{\lambda})[a(q - \hat{\lambda}, \tfrac{1}{2}) + a(q + \hat{\lambda}, \tfrac{1}{2})] \tag{99}$$

$$\int \mathrm{d}q\, \rho(q)[a(\hat{\lambda} - q, \tfrac{1}{2}) + a(\hat{\lambda} + q, \tfrac{1}{2})] \\ = 2\hat{\sigma}(\hat{\lambda}) + 2\hat{\sigma}^h(\hat{\lambda}) + \int \mathrm{d}\hat{\lambda}'\, \hat{\sigma}(\hat{\lambda}')[a(\hat{\lambda} - \hat{\lambda}', 1) + a(\hat{\lambda} + \hat{\lambda}', 1)] \tag{100}$$

if we set $G \to +\infty$. By Fourier transformation of equation (98) we obtain that

$$\int \mathrm{d}q\, \rho(q) + 2\int \mathrm{d}\lambda\, \sigma(\lambda) + \int \mathrm{d}\lambda\, \sigma^h(\lambda) = 1$$

which gives that $N/G = 1 - \int \mathrm{d}\lambda\, \sigma^h(\lambda)$. Owing to $\sigma^h(\lambda) \geqslant 0$, we have that $N \leqslant G$, which coincides with the single occupancy of every site. We assume that there is one particle per lattice site, that is, $N/G = 1$. Then we have $\sigma^h(\lambda) = 0$. Now we consider the case of non-magnetic $\rho(q) = 0$. Relation (98) turns into

$$a(\lambda, 1) = \sigma(\lambda) + \int \mathrm{d}\lambda'\, \sigma(\lambda') a(\lambda - \lambda', 1). \tag{101}$$

By Fourier transformation of equation (100) we have that $\hat{\sigma}(\hat{\lambda}) = \hat{\sigma}^h(\hat{\lambda}) = 0$, which means that $S_z/G = 0$ and the system is non-magnetic. From the above relation, we have that

$$\sigma(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega\lambda} \frac{\mathrm{e}^{-\frac{|\omega|}{2}}}{2\cosh\frac{\omega}{2}}\, \mathrm{d}\omega. \tag{102}$$

The interesting thing is that the above expression is identical to the integrable narrow-band model with periodic boundary obtained by Schlottmann [22]. In this way, relation (99) reduces to

$$a(q, \tfrac{1}{2}) = \rho^h(q) + \int \mathrm{d}\lambda\, \sigma(\lambda) a(q - \lambda, \tfrac{1}{2})$$

and it gives that

$$\rho^h(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega q}}{2\cosh\frac{\omega}{2}}\, \mathrm{d}\omega = \begin{cases} \frac{1}{2}\sec h|\pi q| & \text{for } q \neq 0 \\ \frac{1}{2} & \text{for } q = 0. \end{cases} \tag{103}$$

The number M of the down spins is equal to $G/2$. The ground-state energy is $E/G = -2\ln 2$ for $J = 2$, $V = -\frac{1}{2}$, which has the same value as the one in the periodic boundary condition [22]. This is why the impurities located at both ends cause only the finite-size correction of the ground-state energy. For the case of $J = -2$, $V = \frac{1}{2}$, corresponding to the ferromagnetic state, we have that

$$\rho(q) = \frac{1}{\pi}\frac{\frac{1}{2}}{q^2 + \frac{1}{4}}$$
$$\hat{\sigma}^h(\lambda) = \frac{1}{\pi}\frac{1}{\lambda^2 + 1} \qquad \hat{\sigma}(\lambda) = 0 \tag{104}$$

by taking into account $\sigma(\lambda) = \sigma^h(\lambda) = 0$. Then we have $E/G = 0$.

6. Finite-size correction of the ground state

We assume that the distribution functions $\sigma(\lambda)$, $\rho(q)$ and $\hat{\sigma}(\hat{\lambda})$ are even functions about parameters λ, q and $\hat{\lambda}$, respectively. Then we have the following equations:

$$a(\lambda, 1) + \frac{1}{2G}[a(\lambda, C_a + \tfrac{3}{2}) + a(\lambda, C_a - \tfrac{1}{2}) + a(\lambda, C_b + \tfrac{3}{2}) + a(\lambda, C_b - \tfrac{1}{2})]$$
$$= \frac{1}{2G}a(\lambda, \tfrac{1}{2}) + \sigma(\lambda) + \sigma^h(\lambda) + \int \mathrm{d}\lambda'\, \sigma(\lambda')a(\lambda - \lambda', 1)$$
$$+ \int \mathrm{d}q\, \rho(q)a(\lambda - q, \tfrac{1}{2}) \tag{105}$$

$$a(q, \tfrac{1}{2}) + \frac{1}{2G}[a(q, C_a + 1) + a(q, C_b + 1)]$$
$$= \rho(q) + \rho^h(q) + \int \mathrm{d}\lambda\, \sigma(\lambda)a(q - \lambda, \tfrac{1}{2}) + \int \mathrm{d}\hat{\lambda}\, \hat{\sigma}(\hat{\lambda})a(q - \hat{\lambda}, \tfrac{1}{2}) \tag{106}$$

$$\frac{1}{2G}[a(\hat{\lambda}, \tfrac{1}{2}) + a(\hat{\lambda}, C_a + \tfrac{1}{2}) + a(\hat{\lambda}, C_b + \tfrac{1}{2}) - a(\hat{\lambda}, C_a - \tfrac{1}{2}) - a(\hat{\lambda}, C_b - \tfrac{1}{2})]$$
$$+ \int \mathrm{d}q\, \rho(q)a(\hat{\lambda} - q, \tfrac{1}{2})$$
$$= \hat{\sigma}(\hat{\lambda}) + \hat{\sigma}^h(\hat{\lambda}) + \int \mathrm{d}\hat{\lambda}'\, \hat{\sigma}(\hat{\lambda}')a(\hat{\lambda} - \hat{\lambda}', 1) \tag{107}$$

from equations (92)–(94), where $a(\lambda, \eta) \equiv \eta/[\pi(\lambda^2 + \eta^2)]$ with the arbitrary real parameter η. The terms with factors $1/G$ in the above three equations describe the finite-size corrections of the system. The energies of the system can be described by

$$\frac{E}{G} = \mp\frac{2N}{G} \pm 2\pi\left[\int \mathrm{d}q\, \rho(q)a(q, \tfrac{1}{2}) + \int \mathrm{d}\lambda\, \sigma(\lambda)a(\lambda, 1)\right] \tag{108}$$

for $J = \pm 2$, $V = \mp\frac{1}{2}$, respectively. Setting

$$S_\eta \equiv \mathrm{sign}(\eta) = \begin{cases} 1 & \eta > 0 \\ -1 & \eta < 0 \\ 0 & \eta = 0 \end{cases} \tag{109}$$

we have that

$$\tilde{a}(\omega, \eta) = S_\eta \exp(-|\omega\eta|). \tag{110}$$

By Fourier transforming equation (105), we have

$$\tilde{\sigma}^h(0) = \frac{1}{2G}[S_{C_a+\frac{3}{2}} + S_{C_a-\frac{1}{2}} + S_{C_b+\frac{3}{2}} + S_{C_b-\frac{1}{2}} - 1]$$

for $N/G = 1$. By letting

$$b(\lambda) = a(\lambda, C_a + \tfrac{3}{2}) + a(\lambda, C_a - \tfrac{1}{2}) + a(\lambda, C_b + \tfrac{3}{2}) + a(\lambda, C_b - \tfrac{1}{2}) - a(\lambda, \tfrac{1}{2}) \qquad (111)$$

we have that

$$\int \mathrm{d}q\, \rho(q) a(q, \tfrac{1}{2}) + \int \mathrm{d}\lambda\, \sigma(\lambda) a(\lambda, 1) = a(0, 1) + \frac{b(0)}{2G} - \sigma(0) - \sigma^h(0).$$

We set $\sigma^h(\lambda) \equiv 0$. The Kondo coupling constants C_a and C_b should be in the ranges (i) $C_a > \frac{1}{2}$, $C_b = -\frac{3}{2}$; (ii) $C_a = \frac{1}{2}$, $\frac{1}{2} > C_b > -\frac{3}{2}$; (iii) $\frac{1}{2} > C_a > -\frac{3}{2}$, $C_b = \frac{1}{2}$; (iv) $C_a = -\frac{3}{2}$, $C_b > \frac{1}{2}$. For the case of $J = 2$, $V = -\frac{1}{2}$, the ground-state energy can be written as the form

$$\frac{E}{G} = \frac{\pi}{G} b(0) - 2\pi\sigma(0) \qquad (112)$$

and $\sigma(0)$ should take its largest value. Then we set $\rho(q) = 0$ and obtain that

$$2\pi\sigma(0) = 2\ln 2 + \frac{1}{2G} \int_{-\infty}^{+\infty} \frac{\tilde{b}(\omega)}{1 + \exp(-|\omega|)}\, \mathrm{d}\omega \qquad (113)$$

where

$$\tilde{b}(\omega) = S_{C_a+\frac{3}{2}} \exp[-|\omega(C_a + \tfrac{3}{2})|] + S_{C_a-\frac{1}{2}} \exp[-|\omega(C_a - \tfrac{1}{2})|]$$
$$+ S_{C_b+\frac{3}{2}} \exp[-|\omega(C_a + \tfrac{3}{2})|] + S_{C_b-\frac{1}{2}} \exp[-|\omega(C_b - \tfrac{1}{2})|] - \exp(-|\tfrac{\omega}{2}|).$$

Therefore, the finite-size correction of the ground-state energy due to impurities is

$$E' = \frac{8(2C_a^2 + 3C_a + 2)}{(2C_a + 3)(4C_a^2 - 1)} - \frac{3}{2} - \ln 2 + \frac{\pi}{2} - 2\beta\left(\frac{2C_a - 1}{2}\right) \qquad (114)$$

when $C_a > \frac{1}{2}$ and $C_b = -\frac{3}{2}$, where β is defined by $\beta(x) = \frac{1}{2}[\psi(\frac{x+1}{2}) - \psi(\frac{x}{2})]$ and $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln\Gamma(x)$. By taking account of $C_a > \frac{1}{2}$, then $J_a < 0$, we have $\hat{\sigma}(\hat{\lambda}) = 0$ for the ground state. From relations (105)–(107), we obtain that

$$\hat{\sigma}^h(\hat{\lambda}) = \frac{1}{2G}[a(\hat{\lambda}, \tfrac{1}{2}) + a(\hat{\lambda}, C_a + \tfrac{1}{2}) + a(\hat{\lambda}, -1) - a(\hat{\lambda}, C_a - \tfrac{1}{2}) - a(\hat{\lambda}, -2)] \qquad (115)$$

$$\rho^h(0) = \frac{1}{2} + \frac{1}{2\pi G}\left[\frac{1}{C_a} - \frac{10}{3} + \ln 2 + \frac{\pi}{2} - 2\beta(C_a)\right] \qquad (116)$$

$$\rho^h(q) = \tfrac{1}{2}\sec h|\pi q| + \frac{1}{2G}[a(q, C_a + 1) - a(q, \tfrac{1}{2})] - \frac{1}{4\pi G}\int_0^{+\infty} \frac{\tilde{b}(\omega)\cos(\omega q)}{\cosh\frac{\omega}{2}}\, \mathrm{d}\omega \qquad (117)$$

for $q \neq 0$. When $C_a = \frac{1}{2}$, $-\frac{3}{2} < C_b < \frac{1}{2}$, from relation (113), we have that

$$2\pi\sigma(0) = 2\ln 2 + \frac{1}{2G}\left\{2(1 - \ln 2) - \pi + 2\left[\beta\left(\frac{2C_b + 3}{2}\right) - \beta\left(\frac{1 - 2C_b}{2}\right)\right]\right\}. \qquad (118)$$

Then, the finite-size correction of the ground state has the form

$$E' = \frac{\pi}{2}[\cot(\tfrac{1}{4}\pi + \tfrac{1}{2}\pi C_b) + \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\pi C_b) + 1]$$
$$+ \ln 2 - \frac{1}{2}\frac{44C_b^2 - 26C_b - 35 + 40C_b^3}{(2C_b + 3)(2C_b - 1)(2C_b + 1)}. \qquad (119)$$

By taking account of $J_b > 0$, we have $\hat{\sigma}^h(\hat{\lambda}) = 0$. From relations (105)–(107), we obtain that

$$\hat{\sigma}(\hat{\lambda}) = \frac{1}{4\pi G}\int_0^{+\infty} \mathrm{d}\omega\,\frac{\cos(\omega\lambda)}{\cosh\frac{\omega}{2}}\left\{1+\exp\left(-\left|\frac{\omega}{2}\right|\right)+S_{C_b+\frac{1}{2}}\exp\left[\left|\frac{\omega}{2}\right|-\left|\omega\left(C_b+\frac{1}{2}\right)\right|\right]\right\} \tag{120}$$

$$\begin{aligned}\rho^h(q) = \tfrac{1}{2}\sec h|\pi q| &+ \frac{1}{2G}[a(q,\tfrac{3}{2})+a(q,C_b+1)] \\ &-\frac{1}{4\pi G}\int_0^{+\infty}\mathrm{d}\omega\,\frac{\cos(\omega q)}{\cosh\frac{\omega}{2}}\{\exp(-|2\omega|)+\exp(-|\omega|)+\exp[-|\omega(C_b+\tfrac{3}{2})|] \\ &+S_{C_b+\frac{1}{2}}\exp[-|\omega(C_b+\tfrac{1}{2})|]\}\end{aligned} \tag{121}$$

for $q \neq 0$ and

$$\rho^h(0) = \begin{cases}\frac{1}{2} & \text{for } C_b = -1 \\ \frac{1}{2}+\frac{1}{2\pi G}\left\{\frac{1}{C_b+1}+\beta(C_b+2)\right. & \\ \qquad\left.+S_{C_b-\frac{1}{2}}\beta\left(\frac{|2C_b+1|+1}{2}\right)\right\} & \text{for } C_b \neq -1.\end{cases} \tag{122}$$

The cases of $\frac{1}{2} > C_a > -\frac{3}{2}$, $C_b = \frac{1}{2}$; $C_a = -\frac{3}{2}$, $C_b > \frac{1}{2}$ have the similar expressions. When $J = -2$, $V = \frac{1}{2}$, by similar discussions, the finite-size correction of the ground-state energy can be written as

$$E' = \frac{5}{2} - \frac{4(2C_a+1)}{(2C_a-1)(2C_a+3)} \tag{123}$$

for $C_a > \frac{1}{2}$, $C_b = -\frac{3}{2}$ and

$$E' = \frac{3}{2} - \frac{4(2C_b+1)}{(2C_b-1)(2C_b+3)} \tag{124}$$

for $C_a = \frac{1}{2}$, $\frac{1}{2} > C_b > -\frac{3}{2}$.

Therefore, an integrable model in one dimension is constructed from the t–J model where two magnetic impurities are coupled to the system. It describes the behaviour of the strong correlation electrons with Kondo problem. The spectra of the system are not linear. The boundary R-matrix depends on the spin and rapidity of the particle and satisfies the reflecting factorizable condition. The Hamiltonian of the model is diagonalized exactly by the Bethe ansatz method. The integral equations are derived with the complex 'rapidities' q which describe the bound states in the system. The properties of the ground state are discussed and the finite-size corrections of the ground-state energies are obtained due to the couplings of the magnetic impurities.

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